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# Discrete Torsion and Symmetric Products

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Abstract

In this note we point out that a symmetric product orbifold CFT can be twisted by a unique nontrivial two-cocycle of the permutation group. This discrete torsion changes the spins and statistics of corresponding second-quantized string theory making it essentially "supersymmetric." The long strings of even length become fermionic (or ghosts), those of odd length bosonic. The partition function and elliptic genus can be described by a sum over stringy spin structures. The usual cubic interaction vertex is odd and nilpotent, so this construction gives rise to a DLCQ string theory with a leading quartic interaction.

## 1. Introduction

Symmetric product orbifolds are two-dimensional sigma models on the configuration space

$$S^N X = X^N / S_N \tag{1.1}$$

of $N$ unordened points on a space $X$. Although this is a singular space, since the permutation group $S_N$ does not act free, the corresponding CFT is well-behaved. In the last years these CFT's have played an important role in string theory. Symmetric products appear naturally as moduli spaces of supersymmetric gauge theories and world-volume theories of D-branes, and the corresponding CFT's are a crucial ingredient in matrix string theory [1, 2, 3] and the light-cone gauge quantization of so-called little string theories [4, 5, 6]. (See also [7] for background material on symmetric product models.) In all these applications one describes second-quantized string theories by considering a single sigma model with target $S^N X$, effectively second-quantizing the spacetime first.

In this note we want to point out that these orbifold conformal field theories have a natural simple generalization. One can include a nontrivial discrete torsion class

$$H^2(S_N, U(1)) = \mathbf{Z}_2 \tag{1.2}$$

for the action of the permutation group. With this torsion the configuration space becomes in some mild sense a non-commutative space. The discrete torsion class defines a spin cover of the permutation group where disjoint elementary transpositions anticommute instead of commute. We will point out that physically the string theory becomes essentially supersymmetric, with equal number of bosonic and fermionic strings, where the statistics are determined by the winding number of the string (with even windings fermionic and odd winding bosonic).

The plan of this note is as follows. We will first recall the definition and some interpretations of discrete torsion in orbifold conformal field theories. Then we will apply this formalism to the symmetric group in section 3, where we explicitely compute the effects of the two-cocycle (1.2). We then turn to the application in CFT and second-quantized string theory in sections 4 and 5. The partition function, and in particular the elliptic genus, is described in section 6. We end with some concluding remarks and speculations.

## 2. Discrete torsion

Consider an orbifold conformal field theory that is obtained by quotienting by a finite group $G$. Then it is well-known that for any non-trivial two-cocycle

$$c \in H^2(G, U(1)) \tag{2.1}$$

in the group cohomology of $G$ we can define a new model by weighting the twisted sectors of the orbifold with a non-trivial phase, the so-called discrete torsion [8]. In the case of a sigma model on a geometric orbifold $X/G$ this discrete torsion can be seen as a particular choice of a flat $B$-field — or "flat gerbe" in modern parlance [9] — on the (possibly singular) target space.

One way to understand discrete torsion is to realize that orbifold quantum field theories can be considered as discrete gauge theories with gauge group $G$. In a Lagrangian formulation the partition function of such an orbifold theory on a Riemann surface $\Sigma$ is obtained by summing over all possible $G$ bundles over $\Sigma$. Topologically these bundles are classified by homotopy classes of maps of $\Sigma$ into the classifying space $BG$, the base space of the universal $G$ bundle. We can identify the group cohomology $H^*(G)$ with the usual cohomology of the classifying space $BG$. In the path-integral we will sum over all maps

$$x\colon\ \Sigma \to BG, \tag{2.2}$$

and in the presence of a discrete torsion class $c \in H^2(BG, U(1))$ we weight the map $x$ by an extra phase factor [10]

$$\langle x^* c, \Sigma \rangle \in U(1). \tag{2.3}$$

Alternatively, in a Hamiltonian formalism the Hilbert space $\mathcal{H}$ of the orbifold decomposes into sectors as

$$\mathcal{H} = \bigoplus_{[g]} \mathcal{H}_g \tag{2.4}$$

Here the superselection sectors $\mathcal{H}_g$ consist of states twisted by $g \in G$ and are labeled by the conjugacy class $[g]$. The twisted states still carry a residual action of the centralizer $C_g$, the subgroup consisting of all elements of $G$ that commute with the twist $g$. Normally, in an orbifold theory one projects in the twisted sector $\mathcal{H}_g$ on the states invariant under $C_g$. But, in the presence of discrete torsion the projection picks out the states that transform as a particular (possibly trivial) one-dimensional representation of $C_g$. This representation of $C_g$, that we will write as $\epsilon(g, \cdot)$, is given in terms of the group 2-cocycle $c(g, h)$ as

$$\epsilon(g,h) = \frac{c(g,h)}{c(h,g)}, \qquad hg = gh. \tag{2.5}$$

One can think of the representation $\epsilon(g, h)$ geometrically as the phase factor that is associated to the (flat) $G$ bundle on the two-torus $T^2$ given by the commuting pair of holonomies $(g, h)$. By this geometric interpretation, or equivalently by the cocycle condition on $c$, the above definition of $\epsilon(g, h)$ is manifest $SL(2, \mathbf{Z})$ invariant.

One concrete way to think about this particular one-dimensional representation of $C_g$ is as follows. A two-cocycle or "Schur multiplier" $c \in H^2(G, U(1))$ defines a central extension $\widehat{G}$ of $G$,

$$1 \to U(1) \to \widehat{G} \to G \to 1. \tag{2.6}$$

If $G$ is a finite group, then

$$Z := H^2(G, U(1)) \cong H^3(G, \mathbf{Z}) \tag{2.7}$$

is a finite abelian group and the central extension can be restricted to a finite extension by $Z$,

$$1 \to Z \to \widehat{G} \to G \to 1. \tag{2.8}$$

Given such a central extension, and a pair of commuting elements $g, h$ in the group $G$, we can lift the pair to elements $\hat{g}, \hat{h}$ in the covering group $\widehat{G}$. Now in this central extension the lifts do no longer necessarily commute, and the commutator of the elements $\hat{g}$ and $\hat{h}$ gives the required one-dimensional representation that takes values in $Z$. That is, we have

$$\epsilon(g,h) = [\hat{g}, \hat{h}] = \hat{g}\,\hat{h}\,\hat{g}^{-1}\hat{h}^{-1} \in Z. \tag{2.9}$$

Note that the commutator is independent of the choices of lifts, since these choices differ, by definition, by a central element, and these central elements cancel out in the commutator.

## 3. Discrete torsion for the symmetric group

Let us now consider the case where the finite group $G$ is taken to be the symmetric group on $N$ elements $S_N$. In this case it is well-known that in the "stable range" $N \geq 4$

$$H^2(S_N, U(1)) \cong \mathbf{Z}_2. \tag{3.1}$$

That is, there is a unique non-trivial central extension of the permutation group

$$1 \to \mathbf{Z}_2 \to \widehat{S}_N \to S_N \to 1. \tag{3.2}$$

(In the case $N < 4$ this extension by $\mathbf{Z}_2$ still exists, but it is trivial when considered as an extension by $U(1)$. For example for $N = 2$ we get the familiar extension $\mathbf{Z}_2 \to \mathbf{Z}_4 \to \mathbf{Z}_2$.)

Geometrically we can think of this double cover as follows. Consider the natural action of $S_N$ on an orthonormal basis of the vector space $\mathbf{R}^N$. This gives an embedding of $S_N$ into $O(N-1)$, the orthogonal group that acts on the hyperplane that contains the $N$ basis vectors. The orthogonal group has a unique $\mathbf{Z}_2$ central extension $Pin(N-1)$. Restricted to the rotation group $SO(N-1)$ it is the usual spin cover $Spin(N-1)$. The restriction of this spin cover to the symmetric group defines the $\mathbf{Z}_2$ central extension that we are looking for.

It should be noted that in parastatistics, where one considers higher-dimensional representations of the permutation group, this central extension plays an important role, since it leads to spinor representation of the statistics group, see *e.g.* [11]. Such representations where first studied by Schur, and play an important role in the statistics of quasiparticles in the Pfaffian $\nu = \frac{1}{2}$ quantum Hall state.

In terms of generators and relations we can define the spin cover $\widehat{S}_N$ as follows [12, 13]. Let us denote the standard generators of $S_N$ by $t_i$ with $i = 1, \ldots, N-1$. Here $t_i$ is the elementary transposition that interchanges the elements $i$ and $i+1$. These generators satisfy the familiar relations

$$\begin{aligned} t_i^2 &= 1, \\ t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, \\ t_i t_j &= t_j t_i, \qquad j > i+1. \end{aligned} \tag{3.3}$$

The central extension is obtained by replacing these generators by their lifts $\hat{t}_i$, adjoining the central element $z$ with $z^2 = 1$ (we will often write $z = -1$) and modifying the above relations to

$$\begin{aligned} \hat{t}_i^2 &= z, \\ \hat{t}_i \hat{t}_{i+1} \hat{t}_i &= \hat{t}_{i+1} \hat{t}_i \hat{t}_{i+1}, \\ \hat{t}_i \hat{t}_j &= z\, \hat{t}_j \hat{t}_i, \qquad j > i+1. \end{aligned} \tag{3.4}$$

Note that the lift of a transposition has order four, and that two transpositions that act disjunctly anticommute. That is because in the spin extension these operations, which are geometrically (Weyl) reflections, are represented by Dirac matrices and therefore anticommute instead of commute. Indeed this last fact is responsible for the non-trivial two-cocycle of $S_N$. It comes from the $\mathbf{Z}_2 \times \mathbf{Z}_2$ subgroup that is generated by interchanging particles 1 and 2 and particles 3 and 4. These generators now satisfy

$$\hat{t}_1 \cdot \hat{t}_3 = -\hat{t}_3 \cdot \hat{t}_1. \tag{3.5}$$

The discrete torsion of the symmetric group is therefore simply the lift of the group cocycle (so familiar to string theorists) that generates

$$H^2(\mathbf{Z}_2 \times \mathbf{Z}_2, U(1)) \cong \mathbf{Z}_2. \tag{3.6}$$

This also explain why the discrete torsion only appears for $N \geq 4$.

Using these generators and relations it is now completely straightforward to compute explicitly the phases $\epsilon(g,h)$ in the twisted sectors. Here we should keep in mind that any element $g \in S_N$ is conjugate to a product of cyclic permutations, that can be labeled by a partition of $N$

$$[g] = (1)^{N_1}(2)^{N_2}\cdots, \qquad \sum_{n \geq 1} n N_n = N. \tag{3.7}$$

The centralizer of such an element $g$ is given by

$$C_g \cong \prod_{n\geq 1} S_{N_n} \ltimes \mathbf{Z}_n^{N_n} \tag{3.8}$$

Since $\epsilon(g,h)$ is a representation of $C_g$, it satisfies

$$\epsilon(g,h_1h_2) = \epsilon(g,h_1)\epsilon(g,h_2). \tag{3.9}$$

Therefore it suffices to compute the phases $\epsilon(g,h)$ for the two specific kinds of elements $h$ that together generate $C_g$:

(1) a generator of the cyclic group $\mathbf{Z}_n$ of order $n$;
(2) an elementary transposition in $S_{N_n}$ that permutes two of those cycles of length $n$.

We will see that the analysis (and the answer) depends critically on the overall signature or parity $|g| = 0,1 \pmod 2$ of the element $g$, that we note can be written as

$$|g| = \sum_{n\geq 1}(n-1)N_n \pmod 2. \tag{3.10}$$

For example, since we have the fundamental result that for $|i-j|>1$

$$\epsilon(t_i,t_j) = [\hat{t}_i,\hat{t}_j] = -1, \tag{3.11}$$

we will often use the fact that, if two elements $g$ and $h$ act disjunctly, they either commute or they anticommute in the extension $\widehat{S}_N$, depending on their signs,

$$\epsilon(g,h) = (-1)^{|g||h|}. \tag{3.12}$$

Case (1). Let us first consider a sector twisted by an element $g$ that contains a cycle of length $n$. Denote the generator of that cycle $k$, with $k^n=e$. So we can write $g = k\cdot g'$ where $g'$ commutes with $k$. Since $g'$ and $k$ act disjunctly we can conclude that

$$\epsilon(g',k) = (-1)^{|g'||k|}, \tag{3.13}$$

and, because $\epsilon(g,k)=\epsilon(g',k)$, we therefore find

$$\epsilon(g,k) = \begin{cases} 1, & n \text{ odd}, \\ (-1)^{|g|-1}, & n \text{ even}. \end{cases} \tag{3.14}$$

We will give an interpretation of this result later.

Case (2). Let us now consider a permutation $x_n$ that interchanges two disjoint cycles of equal length $n \geq 2$ in $g$. We denote the generators of these two cyclic group as $k'$ and $k''$ and write $k = k' \cdot k''$. We first compute the phase $\epsilon(x_n, k)$. The only non-trivial computation is actually the case $n = 2$, since we will see that the general case follows directly from this.

In the case $n = 2$ we can choose $k' = t_1$ and $k'' = t_3$. The exchange operator $x_2$ satisfies by definition

$$x_2 \cdot t_1 = t_3 \cdot x_2, \tag{3.15}$$

and can be written as

$$x_2 = t_2 t_1 t_3 t_2. \tag{3.16}$$

With some simple algebra one now finds with $k = t_1 t_3$

$$\begin{aligned} \hat{k} \cdot \hat{x}_2 &= \ \ \hat{t}_1\hat{t}_3\hat{t}_2\hat{t}_1\hat{t}_3\hat{t}_2 = -\hat{t}_3\hat{t}_1\hat{t}_2\hat{t}_1\hat{t}_3\hat{t}_2 \\ &= -\hat{t}_3\hat{t}_2\hat{t}_1\hat{t}_2\hat{t}_3\hat{t}_2 = -\hat{t}_3\hat{t}_2\hat{t}_1\hat{t}_3\hat{t}_2\hat{t}_2 \\ &= \ \ \hat{t}_3\hat{t}_2\hat{t}_3\hat{t}_1\hat{t}_2\hat{t}_2 = \ \ \hat{t}_2\hat{t}_3\hat{t}_2\hat{t}_1\hat{t}_2\hat{t}_2 \\ &= \ \ \hat{t}_2\hat{t}_3\hat{t}_1\hat{t}_2\hat{t}_1\hat{t}_2 = -\hat{t}_2\hat{t}_1\hat{t}_3\hat{t}_2\hat{t}_1\hat{t}_2 = -\hat{x}_2 \cdot \hat{k}, \end{aligned} \tag{3.17}$$

so that

$$\epsilon(x_2, k) = -1. \tag{3.18}$$

Since $g$ can be written as $g = g' \cdot k$, where $g'$ acts disjunctly from $x_2$, and since $x_2$ has even parity, we find

$$\epsilon(x_2, g) = \epsilon(x_2, k) = -1. \tag{3.19}$$

This result can be formulated as that with discrete torsion two cycles of length 2 anti-commute instead of commute.

In the general case, where we consider an exchange of two cycles of length $n \geq 2$, we simply should observe that the element $x_n$ has conjugacy class $(2)^n$. The element $k$ acts by a cyclic permutation of length $n$ on these $n$ two-cycles. That is, $k$ is a product of $n-1$ elements that each exchange a pair of two-cycles and therefore are conjugated to $x_2$. Therefore we find

$$\epsilon(x_n, g) = \epsilon(x_n, k) = (-1)^{n-1}. \tag{3.20}$$

This has a straightforward interpretation. When we exchange two $n$-cycles we obtain an extra minus sign if $n$ is even. So $n$-cycles behave as "bosons" if $n$ is odd, and as "fermions" if $n$ is even.

## 4. The CFT interpretation

We will now apply the above computations to determine the effect of discrete torsion on the symmetric product orbifold conformal field theory.

### *4.1. Symmetric product orbifolds*

Suppose we start with a conformal field theory $X$ with Hilbert space $\mathcal{H} = \mathcal{H}(X)$. We want to determine the Hilbert space of the symmetric orbifold $S^N X = X^N/S_N$. In the case without discrete torsion the answer has been determined in [14]. (See also [15] for details about the chiral structure such as fusion rules, modular transformations and braiding matrices of rational permutation orbifold CFT's.)

The answer is most elegantly formulated if one considers the direct sum of all symmetric products $S^N X$ summed over $N$. Then the resulting Hilbert space has the structure of a Fock space, generated by an infinite set of Hilbert spaces $\mathcal{H}_n$, $n \geq 1$, based on the target space $X$.

More precisely, if we use the formal generating space of symmetric powers with dummy variable $p$

$$S_p \mathcal{H} = \sum_{N \geq 0} p^N S^N \mathcal{H}, \tag{4.1}$$

(and similarly $\bigwedge_p \mathcal{H}$ for the generating space of exterior powers), then we have the result

$$\sum_{N \geq 0} p^N \mathcal{H}(S^N X) = \bigotimes_{n>0} S_{p^n} \mathcal{H}_n(X), \tag{4.2}$$

where the Hilbert spaces $\mathcal{H}_n(X)$ are defined as the subspace of $\mathcal{H}(X)$ satisfying

$$L_0 - \overline{L}_0 = 0 \ (\mathrm{mod}\ n). \tag{4.3}$$

The Hilbert spaces $\mathcal{H}_n$ carry redefined Hamiltonians $L_0^{(n)} = L_0/n$. One refers to these states generally as "long strings" [16] of length $n$.

Note that the sector $\mathcal{H}_g$ of the orbifold $S^N X$ twisted by a group element $g$ of conjugacy class

$$[g] = \prod_{n \geq 1} (n)^{N_n} \tag{4.4}$$

corresponds to the summand

$$\bigotimes_{n \geq 1} S^{N_n} \mathcal{H}_n \tag{4.5}$$

in the above expression.

In all these formulas it should be remembered that our starting point is an action of the symmetric group on $S^N\mathcal{H}$. Here we have some freedom in how we want $S_N$ to act. In general, the space $\mathcal{H}$ will be graded and the corresponding action of $S_N$ will respect this gradation. That is, it will act in the appropriate way by symmetrization or anti-symmetrization on the even or odd parts. For example if $\mathcal{H}$ is the Hilbert space of a free fermion, it will be $\mathbf{Z}_2$ graded by the fermion number. All actions of the permutation group are always assumed to be in this graded sense.

### *4.2. Symmetric products with torsion*

Let us now see how these formulas are changed if one includes discrete torsion. First of all we have to distinguish between twisted sectors related to $n$-cycles with $n$ odd and even. According to our formula (3.20) the statistics of these sectors is now commuting or anticommuting depending on whether $n$ is odd or even. We will refer to these states as bosons and fermions. The "fermion number" $F$ (mod 2) of a twisted state simply equals the sign of the corresponding cyclic permutation. The total parity $|g|$ of the twisted sector $g$ thus reflects the total fermion number of the state. Note that this combinatorial fermion number should be added to the fermion number that might already be present in the gradation of the one-string Hilbert space $\mathcal{H}(X)$.

Secondly, for a sector corresponding to a $n$-cycle we also have to implement the $\mathbf{Z}_n$ projection. As we have seen, in the case without discrete torsion invariance under the action of this $\mathbf{Z}_n$ subgroup changes the conventional level matching condition. Instead of the requirement that the spin $L_0 - \overline{L}_0$ is integer, one now restricts to the modified condition that in the Hilbert space $\mathcal{H}_n$ the spin is an $n$-fold as in (4.3).

With discrete torsion turned on, formula (3.14) tells us that we have to project differently in the case that $n$ is even and $(-1)^{|g|} = 1$. In that case we no longer require invariance under $\mathbf{Z}_n$ but instead only keep the states in the sign representation where the generator of $\mathbf{Z}_n$ is represented as $-1$. So the new level matching condition gives the following quantization for the spin $m$ of strings of even length $n$:

$$m := L_0^{(n)} - \overline{L}_0^{(n)} = \frac{1}{n}(L_0 - \overline{L}_0) \in \mathbf{Z} + \tfrac{1}{2}. \tag{4.6}$$

That is, in the orbifold CFT the "fermionic" states of even length $n$ will have half-integer spin. Since $L_0 - \overline{L}_0$ is always an integer, this condition clearly only makes sense for $n$ even. Note that the overall spin of this state is still integer, because we only apply this construction in the case that the total fermion number is even. We will denote quite generally as $\mathcal{H}_n^{\pm}$ the subspaces of $\mathcal{H}_n$ of (half)integer spin, *i.e.*, the subspaces consisting of states that satisfy $(-1)^{2m} = \pm 1$.

Adding everything together the Hilbert space of the symmetric product orbifold $S^N X$

with the nontrivial 2-cocycle $c \in H^2(S_N, U(1))$ takes the following form

$$\mathcal{H}^c(S^N X) = \bigoplus_{\substack{even\,\{N_n\} \\ \sum nN_n = N}} \bigotimes_{n>0} S_{p^{2n-1}} \mathcal{H}^+_{2n-1} \otimes \textstyle\bigwedge_{p^{2n}} \mathcal{H}^-_{2n}$$

$$\bigoplus_{\substack{odd\,\{N_n\} \\ \sum nN_n = N}} \bigotimes_{n>0} S_{p^{2n-1}} \mathcal{H}^+_{2n-1} \otimes \textstyle\bigwedge_{p^{2n}} \mathcal{H}^+_{2n} \tag{4.7}$$

Here a partition $\{N_n\}$ is called even or odd depending on the parity $|g|$ of the corresponding permutation.

## 5. Second-quantized strings

Let us make some remarks on the physical interpretation of this result in string theory. Of course the most obvious effect is that the discrete torsion has changed the statistics of the strings. The "long strings" of even length have turned into fermions, while the strings of odd length are still bosons. Since we roughly have as many bosons as fermions the model looks "supersymmetric," although, as we will point out, it is perhaps better to refer to ghosts instead of fermions. The second, more subtle effect of discrete torsion is its influence on the quantization of the spins of the fermionic strings. This interpretation is made more precise in terms of the light-cone quantization formalism.

### *5.1. Interpretation in DLCQ*

Recall that permutation orbifold CFTs on the configuration space $S^N X$ appear naturally in the description of second-quantized strings theories on

$$X \times \mathbf{R}^{1,1} \tag{5.1}$$

in light-cone gauge, where the longtudinal momentum $p^+$ is discretized, the so-called discrete light-come quantization or DLCQ. In this quantization scheme the null coordinate $x^+$ is used to describe the time evolution of the system, whereas the null coordinate $x^-$, that is canonically conjugated to $p^+$, is assumed to be periodic. This interpretation of the symmetric product orbifold is used in the matrix string interpretation of the ten-dimensional Type IIA string and in the six-dimensional little string theories related to coinciding fivebranes. (See also [17] for a review of little string theories.)

In this interpretation the quantum number $n$, the length of the string, is proportional to the discrete momentum $p^+$. In fact, for our purposes it is best to identify it as

$$p^+ = \frac{n}{2} \tag{5.2}$$

We want to think of the states with $p^+$ integer or half-integer as fields that are periodic or anti-periodic around the compact direction $x^-$. This is a well-known setup for the best-known example of a DLCQ field theory: a chiral conformal field theory in $1+1$ dimensions. Here the momentum $p^+$ is simply the eigenvalue of the operator $L_0$ (not to be confused with the world-sheet $L_0$ used above) and we know that for fermionic field this eigenvalue will be half-integer in the NS spin structure and integer in the R spin structure.

However, something strange is going on here. According to our computation it is the states with $p^+$ integer that have turned into fermions, whereas the bosons have half-integer $p^+$ respectively. Since it is the bosons that are sensitive to the spin structure and get naturally anti-periodic boundary conditions, we see that from this point of view we are actually dealing with fields that are more appropriately called ghosts, since the usual relation between spin and statistics is not satisfied. Because of the clash with the spin-statistics theorem two bosons can decay into a fermion *etc.*

To understand the second effect of discrete torsion we should remember that strings in DLCQ carry two natural quantum numbers: the longitudinal momentum $p^+ = n/2$ discussed above together with a longitudinal winding number $w^-$, an integer that measures the number of times a string is wrapped around the compactified $x^-$ direction. In light-cone gauge the coordinate $x^-$ is expressed in terms of the world-sheet stress-tensor as

$$\partial x^- = \frac{1}{p^+} T(z). \tag{5.3}$$

Therefore the winding number is given by

$$w^- = \frac{1}{2\pi} \oint dx^- = \frac{1}{p^+}(L_0 - \overline{L}_0) = 2m, \tag{5.4}$$

with $m$ the conformal field theory spin. We have seen that $m$ can be half-integer, so $w^-$ can be even or odd. But the combination

$$p^+ w^- = L_0 - \overline{L}_0 = nm \tag{5.5}$$

is always integer, as is required by level-matching.

So in the presence of discrete torsion the fermionic strings, with $n$ even, can have different excitations depending on whether the total fermion number is even or odd. If the total fermion number is even, these fermionic strings have half-integer spin excitations, or equivalently odd winding numbers. If the total fermion number is odd, all winding numbers are even, as is always the case for the bosonic strings.

This interpretation is still not very satisfying nor intuitive, so we will now turn to the dual picture obtained by performing a T-duality.

*5.2. T-duality and stringy spin structures*

There is an interesting point concerning T-duality in the null direction $x^-$. Such a duality will interchange the momentum $p^+$ and the winding number $w^-$, or equivalently the quantum numbers $n$ and $m$. In the case without torsion this is a manifest symmetry of the second-quantized theory. (In order to make this a full symmetry one has to add the $p^+ = 0$ sector, which is always problematic in light-cone quantization. See also the discussion in [18].) This symmetry between momentum and winding modes seems to be broken in the presence of the discrete torsion, since it is momentum $p^+$ that determines the statistics of the strings.

In fact, the complete physical interpretation becomes more conventional if we work in this T-dual framework. If we interchange (to be precise) $p^+$ and $w^-/2$, we have

$$w^- = n, \qquad p^+ = m, \tag{5.6}$$

and strings that are wrapped an even number of times are fermions, and those that are wrapped an odd number of times are bosons. It would be obvious to assume that unwrapped strings with $n = 0$ are also fermionic, although they do not appear explicitly in this DLCQ scheme.

Since we have fermionic fields, one can think of making a spin projection of the model, by summing over all spin structures along the $x^-$ direction and projecting on even fermion number — a procedure implemented by summing over spin structures in the time direction $x^+$. This would be the spacetime equivalent of the familiar procedure in two dimensional CFT, where the sum over spin structures produces out of a fermionic non-local model a bosonic local model. However, introducing spacetime spin structures in string theory introduces an extra complication, because fermions can also carry winding numbers.

Indeed, consider the general situation of a string on a spacetime that contains a $S^1$, say for convenience of radius one.* On this circle we have momenta $p \in \mathbf{Z}$ and winding numbers $w \in \mathbf{Z}$. Level-matching of the CFT will always require that

$$p \cdot w \in \mathbf{Z}. \tag{5.7}$$

Consider now a (spacetime) fermion mode of the string. We have to pick a spin structure on the circle that determines the boundary conditions for such a field. There are two choices: the anti-periodic (Neveu-Schwarz) spin structure that gives $p \in \mathbf{Z} + \frac{1}{2}$ and the periodic (Ramond) spin structure that quantizes the momentum as $p \in \mathbf{Z}$.

However, if the string also carries a winding number $w$ the story gets a bit more complicated. For half-integer $p$ we cannot allow arbitrary integer winding number, since level-matching requires $p \cdot w$ to be integer. So we see that only string states with *even* winding number can be fermions. This is the effect we have been observing in this note: fermion statistics is only consistent for strings of even length. Only they can naturally be anti-periodic.

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*See also the closely related discussion in [19].

Now that we have seen that the even windings can couple to a spin structure, it is naturally to sum over spin structures, both in the compactified space direction (which might be null, as in DLCQ) as in the time direction. The latter procedure has the familiar interpretation as a projection on even fermion number. In the NS sector, where $p$ is half-integer quantized and where there is a unique ground state, this implies that the total parity $|g|$ should be even. This effect we have seen. Only for $|g|$ even could we have half-integer $m$. In the R sector the spin projection is ambiguous, because of the assignment of fermion number to the ground state. According to our formulas we should assign to the ground state odd fermion number, since we require that the total parity $|g|$ is odd in this case.

## 6. The elliptic genus

It is interesting to translate these considerations into a concrete formula for the genus one partition function for the orbifold theory including the effect of the discrete torsion. This formula becomes particularly simple if we restrict to the chiral partition function, the so-called elliptic genus which for Calabi-Yau sigma models with (at least) $\mathcal{N} = 2$ supersymmetry is defined as the following character in the RR sector

$$\chi(X; q, y) = \operatorname{Tr}_{\mathcal{H}_{RR}} \left[ (-1)^F y^{F_L} q^{L_0 - \frac{c}{24}} \right] \tag{6.1}$$

Given the Fourier expansion

$$\chi(X; q, y) = \sum_{m,\ell} c(m, \ell) q^m y^\ell \tag{6.2}$$

of the chiral partition function of the sigma model on $X$, the generating function of the elliptic genera of the symmetric products is

$$Z(p, q, y) = \sum_{N \geq 0} p^N \chi(S^N X; q, y) = \prod_{n>0,\, m,\ell} (1 - p^n q^m y^\ell)^{-c(nm,\ell)} \tag{6.3}$$

As we have sketched in the previous section, the partition function (and therefore also the elliptic genus) for the orbifold with discrete torsion is best written as a sum over spacetime spin structures

$$Z(p, q, y) = \frac{1}{2} \left( Z_{++} + Z_{+-} + Z_{-+} + Z_{--} \right). \tag{6.4}$$

Here the contributions of the four spin structures are

$$
\begin{aligned}
Z_{++}(p,q,y) &= \prod_{n>0,\,m} \frac{\left(1+p^{2n}q^{m-\frac{1}{2}}y^{\ell}\right)^{c(n(2m-1),\ell)}}{\left(1-p^{2n-1}q^{m}y^{\ell}\right)^{c((2n-1)m,\ell)}} \\
Z_{+-}(p,q,y) &= \prod_{n>0,\,m} \frac{\left(1-p^{2n}q^{m-\frac{1}{2}}y^{\ell}\right)^{c(n(2m-1),\ell)}}{\left(1-p^{2n-1}q^{m}y^{\ell}\right)^{c((2n-1)m,\ell)}} \\
Z_{-+}(p,q,y) &= \prod_{n>0,\,m} \frac{\left(1+p^{2n}q^{m}y^{\ell}\right)^{c(2nm,\ell)}}{\left(1-p^{2n-1}q^{m}y^{\ell}\right)^{c((2n-1)m,\ell)}} \\
Z_{--}(p,q,y) &= -\prod_{n>0,\,m} \frac{\left(1-p^{2n}q^{m}y^{\ell}\right)^{c(2nm,\ell)}}{\left(1-p^{2n-1}q^{m}y^{\ell}\right)^{c((2n-1)m,\ell)}}
\end{aligned}
\tag{6.5}
$$

In the case with zero discrete torsion, after a so-called automorphic correction that adds in the $p^+ = 0$ sector and the shift in the ground state energy, the partition function takes the form

$$
\Phi(p,q,y) = (pq)^{-\chi/24} \prod_{n,m,\ell\geq 0} (1-p^n q^m y^{\ell})^{-c(nm,\ell)}. \tag{6.6}
$$

The infinite product $\Phi$ will be typically an automorphic form of the T-duality group $SO(2,3,\mathbf{Z}) \cong Sp(4,\mathbf{Z})$ [20, 21, 18, 22, 23]. This is an example of the famous lifting of a modular form to an automorphic product as discussed by Borcherds [24].

It would be interesting to investigate the automorphic properties of the infinite products that are obtained by including the effect of discrete torsion. Only for a given spin structure the partition function can be computed as a one loop amplitude with target space

$$
X \times T^2 \tag{6.7}
$$

as in [20]. So automorphicity, which is simply the T-duality associated to the light-cone torus $T^2$, is not as straightforward to check. Notice in this respect that the proper T-duality that interchanges $p$ and $q$ seems to be differently realized in this fermionic model. For example, the partition function $Z_{+-}$ transforms under the transformation $p \leftrightarrow q^{\frac{1}{2}}$ as

$$
\log Z_{+-}(q^{\frac{1}{2}}, p^2, y) = -\log Z_{+-}(p,q,y) \tag{6.8}
$$

which suggests that we have some nontrivial multiplier for the free energy $\log Z$. It would also be interesting to know if these second-quantized elliptic genera are naturally related to characters of super-Lie algebras.

If we put $y = 1$ only ground states states with $L_0 = c/24$ contribute, and the elliptic genus degerates to the Euler number or Witten index. For the symmetric product this gives the well-known identity [25, 26, 27, 7]

$$\sum_{N \geq 0} p^N \chi(S^N X) = \prod_{n>0} (1 - p^n)^{-c} \tag{6.9}$$

with $c = \chi(X)$ the Euler number of $X$. This is almost a modular form for the $SL(2, \mathbf{Z})$ action on $\sigma$, with $p = e^{2\pi i \sigma}$. With discrete torsion only the configurations with no or an odd number of strings of even length contribute, so we get instead the expression

$$\prod_{n>0} (1 - p^{2n-1})^{-c} \left[ 1 + \tfrac{1}{2} \prod_{n>0} (1 + p^{2n})^c - \tfrac{1}{2} \prod_{n>0} (1 - p^{2n})^c \right] \tag{6.10}$$

No obvious modular properties remain.

## 7. Concluding remarks

Matrix string theory can be seen as a DLCQ version of string field theory. It describes perturbative string theory by conformal perturbation theory around the orbifold CFT $S^N \mathbf{R}^8$, where the leading irrelevant operator can be identified with the Mandelstam cubic string vertex [3]. If we twist the CFT with our discrete torsion we obtain a model with exotic statistics. In the large $N$ limit the momentum $p^+ = n/2$ is send to infinity, keeping the ratio $p^+/N$ finite and making it a continuous variable. The distinction between even and odd $n$ disappears, and one gets truly equal number of bosons and fermions.

However, a cubic string vertex, which is realized as a $\mathbf{Z}_2$ twist field, now has fermionic statistics. That is, the standard string coupling constant $g_s$ becomes a *nilpotent* Grassmann variable that squares to zero

$$g_s^2 = 0, \tag{7.1}$$

a rather mystifying phenomenon. Because this cubic vertex is itself fermionic, it mediates interactions where two fermionic strings (say of length 2) can combine to another fermionic string (of length 4).

The next to leading order perturbation is a $\mathbf{Z}_3$ twist field. This represents in string perturbation a quartic contact term. Its couping constant (which in the conventional setup is proportional to $g_s^2$) remains bosonic and is now the leading irrelevant deformation.

The inclusion of discrete torsion is also interesting in the case of the so-called D1-D5 system in Type IIB string theory compactified on a four-torus or $K3$ manifold $X$. Such a D1-D5-brane represents a string in the remaining six uncompactified dimensions. The infrared limit of the world-sheet theory gives rise to a $\mathcal{N} = (4, 4)$ SCFT on the moduli space of instantons on $X$. The central charge is given by $c = 6k$ with $k = Q_1Q_5$, the product of the number of D1-branes and D5-branes. The number of real marginal deformations of this SCFT is $4 \times 5$ or $4 \times 21$ for $X = T^4$ or $K3$ respectively.

For certain values of the space-time moduli this hyperkähler moduli space coincides with the symmetric product $S^NX$ [5, 28]. In these points the addition of discrete torsion gives a different component of the moduli space of $\mathcal{N} = (4,4)$ SCFT. The marginal deformation away from the symmetric product is a $\mathbf{Z}_2$ twist field. Discrete torsion removes this marginal operator. The singularities get frozen in. This component of the $c = 6k$ $\mathcal{N} = (4,4)$ SCFT moduli space is therefore described just by the moduli of $X$. In particular there is no way to deform the model to a regime where the weakly coupled supergravity approximation of the dual formulation as string theory on $AdS_3 \times S^3 \times X$ makes sense.

One of the striking properties of the relation between space-time physics and the D1-D5 CFT is that the elusive Ramond-Ramond gauge fields appear as more traditional $B$-fields in the sigma model. Since we are discussing here a discrete $B$-field on the symmetric product, the spacetime interpretation would seem to be some new discrete RR flux in the $AdS_3 \times S^3 \times X$ string theory. It would be very interesting to identify directly this RR flux.

For orbifold CFT with gauge group $G$ there are natural interpretations of the cohomology groups $H^i(G, U(1))$ for $i = 1,2,3$. The group $H^1(G)$ labels the one-dimensional representations and can be used to twist the original $G$ action. For the permutation group we have $H^1(S_N) = \mathbf{Z}_2$ and this just means that we can choose the short strings to be either fermionic or bosonic.

We have discussed at length the effect of $H^2(S_N)$, how it effects the statistics of the long strings. So this leaves the possible interpretation of the cohomology group $H^3(S_N)$. For a general orbifold it classifies the possible chiral structures of the CFT. Alteratively, it gives the possible three-dimensional topological discrete Chern-Simons gauge theories with gauge group $S_N$ [10]. It is an interesting fact that for the symmetric group there is a well-known factor

$$\mathbf{Z}_{24} \subset H^3(S_N, U(1)). \tag{7.2}$$

As far as I understood, this occurrence of the number 24 is directly related to the famous $c/24$ in CFT, the framing ambiguity in three-manifold invariants, and the Euler number of $K3$. It would be fascinating to know if it has any application in terms of second-quantized strings.

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